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Exact S-Matrices for Bound States of $a_2^{(1)}$ Affine Toda Solitons

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ABSTRACT

Using Hollowood's conjecture for the S-matrix for elementary solitons in complex $a_n^{(1)}$ affine Toda field theories we examine the interactions of bound states of solitons in $a_2^{(1)}$ theory. The elementary solitons can form two different kinds of bound states: scalar bound states (the so-called breathers), and excited solitons, which are bound states with non-zero topological charge. We give explicit expressions of all S-matrix elements involving the scattering of breathers and excited solitons and examine their pole structure in detail. It is shown how the poles can be explained in terms of on-shell diagrams, several of which involve a generalized Coleman-Thun mechanism.

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1 Introduction

This paper deals with a certain class of so-called two-dimensional completely integrable massive models, which are relativistic quantum field theories defined in 1+1 dimensional Minkowski space. Two-dimensional models were first studied during the seventies and have attracted increased attention during the last fifteen years. Zamolodchikov [1] was the first to notice that integrable theories can be obtained as special deformations of conformal field theories, where due to the deformation the conformality is destroyed, but the integrability of the theory can be preserved and the particles acquire masses. From the viewpoint of integrable theories, conformal field theories can be regarded as limits of certain massive integrable models and appear as their particular massless limits.

In order to obtain all the on-shell information of a particular quantum field theory we seek to find an explicit expression for the S-matrix. However, finding exact solutions and understanding non-perturbative effects in quantum field theories in higher dimensions is an extremely difficult task and is regarded as one of the outstanding problems of theoretical physics. In quantum field theories defined in 1+1 dimensions, however, this problem appears to be more manageable. In most cases it is possible to give an explicit expression for the S-matrix, which is both self-consistent and consistent with a Lagrangian formulation of the theory. The main tool for finding exact S-matrices is the exact S-matrix program, which was developed in the sixties in order to describe the strong interaction (see [2]). In the case of integrable theories the axioms of exact S-matrix theory, which will be discussed in section 2, enable us in many cases to find explicit expressions for the S-matrices up to a so-called CDD-factor.

In this paper we consider a special class of integrable two-dimensional theories, the so-called affine Toda field theories (ATFTs). Affine Toda field theories are a family of integrable quantum field theories in 1+1 dimensional Minkowski space. They are defined by their Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - \frac{m^2}{\tilde{\beta}^2}\sum_{j=0}^{r} n_j e^{\tilde{\beta}\alpha_j \phi}. \tag{1}$$

where $\phi = (\phi^1, ..., \phi^r)$ is a r-dimensional scalar field, the α_i are a set of $r+1$ r-dimensional vectors and m and $\tilde{\beta}$ are mass and coupling constant parameters. It can be shown that the theories are integrable if $(\alpha_1, ..., \alpha_r)$ form the root system of a semi-simple classical (rank r) Lie algebra g_r (the longest root is taken to have length $\sqrt{2}$). We chose $\alpha_0 = -\sum_{j=1}^{r} n_j \alpha_j$ to be the negative of the longest root (in case of $a_n^{(1)}$ all n_j are equal to 1), such that the system $(\alpha_0, ..., \alpha_r)$ corresponds to the extended Dynkin diagram of the affine Kac-Moody algebra $\hat{g}_r$. Thus, there is one ATFT associated with each affine Kac-Moody algebra $\hat{g}_r$.

One has to distinguish between the two fundamentally different cases of real and complex ATFT (i.e. with real or imaginary coupling constant $\tilde{\beta}$). In the case of real coupling constant, the Lagrangian defines a unitary quantum field theory containing r particles. Exact S-matrices for the simply laced ATFTs, i.e. based on the a, d and e series of Lie algebras, have been suggested by Braden et al. in [3] and independently by Christe and

Mussardo in [4]. These conjectures have also been shown to be consistent with the results of perturbation theory to low orders in $\tilde{\beta}$. S-matrices for the non-simply laced (b, c, f and g algebras) as well as for the twisted cases, in which the poles corresponding to bound states need to have an additional coupling constant dependence, have been found by Delius et al. in [5]. Whereas the real case is fairly well understood, a complete understanding of the complex ATFTs remains elusive.

The complex ATFTs (the coupling constant is purely imaginary: $\tilde{\beta} = i\beta$ and β real) are far more complicated, but have also proved more interesting. Apart from complex $a_1^{(1)}$ ATFT, which is the well known Sine-Gordon theory (SG), these theories are in general non-unitary. The most striking feature of complex ATFTs is the fact that they admit classical soliton solutions. For the simplest case, the Sine-Gordon theory, these soliton solutions have been known for a long time. Only recently, however, have soliton solutions been found for other ATFTs. It can easily be seen that in the complex case the field is periodic with respect to the weight lattice Λ^* of g_r. Since the original Lagrangian was chosen in a way that $\phi = 0$ was the zero-energy solution, the constant fields $\phi = \frac{2\pi}{\beta}\omega$ (with $\omega \in \Lambda^*$) all have zero energy and thus, unlike in the real case, the vacuum of the complex theory is degenerate. This fact suggests already the existence of soliton solutions of the classical equations of motion, which interpolate between different vacua. For each soliton solution the difference of the field between the two vacua is called the topological charge t of the solution. The topological charge is also an element of the weight lattice Λ^* and can be written as:

$$t = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dx \partial_x \phi \quad , \quad \in \Lambda^* .$$

Explicit expressions for one-soliton and multi-soliton solutions in $a_n^{(1)}$ ATFT were first found in [6] by using Hirota's method for the solution of nonlinear differential equations. Hollowood was also able to show that despite the complex form of the Hamiltonian the solitons have real and positive energies and masses. (For a more detailed discussion of the reality problem of affine Toda solitons see references [7],[8].) More one-soliton and multi-soliton solutions for other algebras have been found in [9],[10] and [11]. By using a different method Olive et al. found a general form for the soliton solutions for all ATFTs in [12].

Apart from the elementary solitons there are also bound states of solitons which can be seen as two solitons oscillating around a fixed point. Whereas in SG theory only bound states with zero topological charge (the 'breathers') occur, in most other ATFTs there are also bound states with non-zero topological charges (the 'breathing solitons' or 'excited solitons'). In [13] classical bound state solutions for $a_n^{(1)}$ were obtained by changing the real velocity into an imaginary velocity in the expressions for the two-soliton solutions. Whereas in the real ATFTs only those r particles exist which are manifest in the Lagrangian, the complex ATFTs contain a rich spectrum of solitons, bound states and excited solitons. A great deal of work has been done on the classical soliton solutions of complex affine Toda field theories and their topological charges but heretofore very little progress has been made towards a consistent quantization of these solutions. In particular, the complete characterisation of the bound-state spectrum for any other quantum ATFT, apart from

Sine-Gordon theory, has been an unsolved problem so far.

The layout of this paper is as follows: Section 2 gives a brief introduction to the axioms of exact S-matrix theory. The Zamolodchikovs' solution of the Sine-Gordon model [14] and Hollowood's S-matrix conjecture for the scattering of solitons in $a_n^{(1)}$ ATFT [15] are reviewed. In section 3 we construct bound states of two elementary solitons. Section 4 forms the main part of this paper, the complete description of $a_2^{(1)}$ ATFT. In this section all S-matrix elements involving the scattering of bound states are constructed explicitly. It is also shown that the lowest scalar bound states can be identified with the fundamental quantum particles. In section 5 the pole structure of these S-matrix elements is discussed and a complete set of three point couplings is conjectured. Using these three-point couplings other poles are explained in terms of higher order diagrams, some of which involve a generalized Coleman-Thun mechanism. Section 6 summarizes the results and concludes with suggestions regarding possible further directions of research.

2 Exact Soliton S-Matrices

In this section we briefly review the axioms of exact S-matrix theory and describe some of the results from references [14] and [15].

2.1 The Axioms of Exact S-Matrix Theory

As mentioned in the introduction, the axioms of exact S-matrix theory strongly constrain the possible form of the S-matrix of an integrable theory. An integrable theory is one in which there exists an infinite number of conserved charges. The existence of these charges implies that any N-particle S-matrix 'factorizes' into a product of $\frac{1}{2}N(N-1)$ two-particle S-matrices. Moreover it can be shown ([14], [16]) that the S-matrices are purely elastic; that is, neither particle creation nor destruction can occur.

Let us assume a theory with particles living in multiplets $V_1, V_2, \ldots, V_{n-1}$, which are complex vector spaces. The two-particle S-matrix $S^{a,b}$ describing the scattering process of two particles lying in V_a and V_b must act as an intertwiner on these spaces:

$$S^{a,b}(\theta_{ab}) : V_a \otimes V_b \to V_b \otimes V_a.$$

(Instead of using momenta, in two dimensions it is convenient to work with the rapidities θ of particles, which are given in terms of the two-momenta: $(p_0, p_1) = m(\cosh|\theta|, \sinh|\theta|)$. The S-matrix is a function of the rapidity difference $\theta_{ab} = \theta_a - \theta_b$ of the incoming particles.) The S-matrix elements $S^{a,b}(\theta)$ have to satisfy the following conditions:

(i) *Unitarity*
If I_a denotes the identity operator on V_a, then the unitary condition can be expressed as

$$S^{a,b}(\theta) S^{b,a}(-\theta) = I_b \otimes I_a,$$

(which does not imply that the Hamiltonian itself is unitary).

(ii) *Crossing Symmetry*
The matrix elements $S^{a,b}(\theta)$ are symmetric under the transformation $\theta \to i\pi - \theta$ such that

$$S^{\overline{a},b}(\theta) = (I_b \otimes C_a)[\sigma S^{b,a}(i\pi - \theta)]^{t_2} \sigma (C_{\overline{a}} \otimes I_b), \tag{2}$$

in which $\overline{a}$ represents the conjugate of particle a. For an exact definition of the charge conjugation operator C_a and the symbols σ and t_2 see reference [15].
(iii) *Analyticity*
$S^{a,b}(\theta)$ is a meromorphic function of θ with the only singularities on the physical strip ($0 \leq \mathrm{Im}\theta \leq \pi$) at $\mathrm{Re}\theta = 0$. The 'physical strip' in the θ-plane corresponds to the physical sheet in the s-plane, where $s = (p_1 + p_2)^2$ is just one of the usual Mandelstam variables. Simple poles in the physical strip correspond to bound states in the direct or crossed channel. (In some cases simple poles could also result from a process called the generalized Coleman-Thun mechanism, which will be discussed in detail in section 5.)
(iv) *Bootstrap*
If $S^{a,b}$ exhibits a simple pole θ^c_{ab} on the physical strip corresponding to a bound state in V_c in the direct channel, then the mass of this particle c is given by the formula:

$$m_c^2 = m_a^2 + m_b^2 + 2 m_a m_b \cos(\mathrm{Im}\theta^c_{ab}) \tag{3}$$

In this case there must also be poles $\theta^{\overline{b}}_{\overline{c}a}$ in $S_{\overline{c}a}$ and $\theta^{\overline{a}}_{b\overline{c}}$ in $S_{b\overline{c}}$, such that $\theta^c_{ab} + \theta^{\overline{b}}_{\overline{c}a} + \theta^{\overline{a}}_{b\overline{c}} = 2\pi i$. The bootstrap equations express the fact that there is no difference whether the scattering process with any particle in, say, V_d occurs before or after the fusion of particles a and b into particle c:

$$[I_b \otimes S^{d,a}(\theta - (i\pi - \theta^{\overline{b}}_{\overline{c}a}))][S^{d,b}(\theta + (i\pi - \theta^{\overline{a}}_{b\overline{c}})) \otimes I_a] = S^{d,c}(\theta) \tag{4}$$

(which is restricted to $V_d \otimes V_c \subset V_d \otimes V_a \otimes V_b$).

The bootstrap equations impose very strong constraints on the exact form of the S-matrix elements and can be used to obtain higher S-matrix elements from the lowest ones (see for instance [3]).

2.2 Sine-Gordon Theory

Only the simplest case of complex ATFTs, that of Sine-Gordon theory ($a_1^{(1)}$-ATFT), has been solved completely by the Zamolodchikovs in [14], where they give explicit expressions for all S-matrix elements describing the scattering of solitons and their bound states. They used a noncommutative algebra to describe the full scattering theory, which we will review briefly here.

Sine-Gordon theory contains two elementary solitons, the fundamental soliton and its anti-soliton, which can be denoted as $A(\theta)$ and $\overline{A}(\theta)$. The possible scattering processes can then formally be written as:

$$\begin{aligned} A(\theta_1)A(\theta_2) &= \tilde{S}^I(\theta_{12})A(\theta_2)A(\theta_1) \\ A(\theta_1)\overline{A}(\theta_2) &= \tilde{S}^T(\theta_{12})\overline{A}(\theta_2)A(\theta_1) + \tilde{S}^R(\theta_{12})A(\theta_2)\overline{A}(\theta_1) \end{aligned} \tag{5}$$

where $\tilde{S}^T(\theta)$, $\tilde{S}^R(\theta)$ and $\tilde{S}^I(\theta)$ denote the S-matrix elements for the transition, reflection and identical particle processes respectively ($\theta_{12} = \theta_1 - \theta_2$ is the rapidity difference of the incoming particles). It should be noted that the reflection process is a purely quantum mechanical effect, and that there is no reflection in the classical scattering of soliton solutions (to discuss the reflection process on the classical level one would need to consider complex time trajectories, see e.g. [15] and references therein).

The above axioms of exact S-matrix theory can be used to obtain exact expressions for the scattering amplitudes (see [14] and references therein). However, for a complete description of the SG theory one has to augment the algebra (5) with symbols $B_p(\theta)$ for the breather states. If a breather B_p corresponds to a pole Θ_p in $S^T(\theta)$ then the Zamolodchikovs formally defined:

$$B_p(\frac{\theta_1+\theta_2}{2}) = \lim_{\theta_2-\theta_1\to\Theta_p} [A(\theta_1)\overline{A}(\theta_2) \pm \overline{A}(\theta_1)A(\theta_2)] \tag{6}$$

(+ if p is even, and $-$ if p is odd).

By considering triple products like $A(\theta_1)\overline{A}(\theta_2)A(\theta_3)$ using (5), (6) and taking the limit, one can calculate explicit expressions of S-matrix elements for the scattering of a breather with a fundamental soliton, or a breather with another breather. (The complete Zamolodchikov algebra and expressions for the S-matrix elements found by the Zamolodchikovs are given in Appendix A.) In section 4 we will use this technique to calculate the bound state scattering amplitudes for $a_2^{(1)}$ ATFT.

2.3 A Quantum Group S-matrix for $a_2^{(1)}$-Solitons

Quantum groups had not yet been introduced in the late seventies when the complete SG S-matrix was found. Only some years later was it discovered that the S-matrix found in [14] has a quantum group symmetry. The requirement of factorizability implies that the two-particle S-matrices satisfy a cubic identity called the 'Yang-Baxter equation', which plays an important role in the theory of quantum groups. Using this fact, Bernard and LeClair in [17] expressed the SG S-matrix in terms of the R-matrix of the quantum group $U_q(Sl(2))$, which is the q-deformed universal enveloping algebra of $Sl(2)$. They also constructed non-local conserved charges for a variety of massive integrable models. In particular they were able to show that for an ATFT based on the algebra g its non-local charges generate the quantum loop algebra of $g^\vee$. This implies that the S-matrix for the scattering of solitons, which is required to commute with the action of these non-local charges, itself must display a quantum group symmetry. This quantum group symmetry of the S-matrix led Hollowood to conjecture $U_q(Sl(n+1))$ symmetric S-matrices for the scattering of solitons in $a_n^{(1)}$ ATFTs [15]. For our purpose of constructing the bound states of $a_2^{(1)}$ ATFT below, we will restrict ourself to the description of Hollowood's construction only for the case of $U_q(Sl(3))$. (For a review of the theory of quantum groups and their representations see for instance [18], [15], [19].)

We will describe the construction of a soliton S-matrix in terms of the $U_q(Sl(3))$ R-matrix, which is a trigonometric solution of the Yang-Baxter-Equation (YBE). In its general form this equation can be written as:

$$
\begin{aligned}
&[\check{R}^{U_2,U_3}(x) \otimes I_{U_1}][I_{U_2} \otimes \check{R}^{U_1,U_3}(xy)][\check{R}^{U_1,U_2}(y) \otimes I_{U_3}] = \\
&= [I_{U_3} \otimes \check{R}^{U_1,U_2}(y)][\check{R}^{U_1,U_3}(xy) \otimes I_{U_2}][I_{U_1} \otimes \check{R}^{U_2,U_3}(x)] \qquad (7)
\end{aligned}
$$

where U_1, U_2 and U_3 are complex vector spaces, $\check{R}^{U_i,U_j} : U_i \otimes U_j \to U_j \otimes U_i$ are linear maps and I_{U_i} is the identity on U_i, thus both sides of (7) map $U_1 \otimes U_2 \otimes U_3$ into $U_3 \otimes U_2 \otimes U_1$ (x is called the spectral parameter).

We are interested in representations associated with the two fundamental representations (ρ_a, V_a) $(a = 1, 2)$ of a_2. (The commonly used notations for ρ_1 and ρ_2 are $\underline{3}$ and $\overline{3}$). Here we can take $V_1 \cong C^3$ and choose (e_1, e_2, e_3) as a basis of V_1. As shown in [18] a basic R-matrix can then be expressed in terms of the generator T of the Hecke-algebra $\mathcal{H}_2$ (which is a generalisation of the symmetric group $\mathcal{S}_2$):

$$\check{R}^{1,1}(x) = xT^{-1} - x^{-1}T$$

and we can use the particular representation

$$
T(e_i \otimes e_j) = \begin{cases} q^{-1} e_i \otimes e_j & \text{if } i = j \\ (q^{-1} - q) e_i \otimes e_j + e_j \otimes e_i & \text{if } i > j \\ e_j \otimes e_i & \text{if } i < j. \end{cases}
$$

It can easily be confirmed that $\check{R}^{1,1}(x) \in \mathrm{End}(V_1 \otimes V_1)$ satisfies the YBE for the special case $U_1 = U_2 = U_3 = V_1$. If we define the basic matrices E_{ij} acting on V_1 by $E_{ij} e_k \equiv \delta_{jk} e_i$ (for $i, j, k = 1, 2, 3$) then we can write explicitly:

$$
\begin{aligned}
\check{R}^{1,1}(x) \;=\; & (xq - x^{-1}q^{-1}) \sum_{i=1}^{3} E_{ii} \otimes E_{ii} + x(q - q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj} \\
& + x^{-1}(q - q^{-1}) \sum_{i>j} E_{ii} \otimes E_{jj} + (x - x^{-1}) \sum_{i \neq j} E_{ji} \otimes E_{ij}. \qquad (8)
\end{aligned}
$$

$\check{R}^{1,1}$ can also be expressed conveniently in terms of the projection operators $s_2^{\pm} = \frac{1}{[2]_q}(q^{\pm 1} \pm T)$, which are the quantum group analogues of the full symmetrizer (antisymmetrizer) and $[n]_q = (q^n - q^{-n})/(q - q^{-1})$,

$$\check{R}^{1,1}(x) = (xq - x^{-1}q^{-1}) s_2^{+} + (x^{-1}q - xq^{-1}) s_2^{-}. \qquad (9)$$

This formula is called the spectral decomposition of $\check{R}^{1,1}$. By applying the fusion procedure [18], we can construct other solutions of the YBE from the basic solution $\check{R}^{1,1}(x)$ in the following way. The module V_2 of the second fundamental representation can be defined as subspace of the tensor product of V_1 with itself by using projector s_2^{-}:

$$V_2 \equiv s_2^{-}(V_1 \otimes V_1)$$

and we define

$$\check{R}^{1,2}(x) = [I \otimes \check{R}^{1,1}(x(-q)^{\frac{1}{2}})][\check{R}^{1,1}(x(-q)^{-\frac{1}{2}}) \otimes I] \tag{10}$$

which acts on $V_1 \otimes V_2$ (I denotes the identity on V_1). In the same way we can define more R-matrices:

$$\begin{aligned} \check{R}^{2,1}(x) &= [\check{R}^{1,1}(x(-q)^{\frac{1}{2}}) \otimes I][I \otimes \check{R}^{1,1}(x(-q)^{-\frac{1}{2}})] \\ \check{R}^{2,2}(x) &= [I \otimes \check{R}^{2,1}(x(-q)^{\frac{1}{2}})][\check{R}^{2,1}(x(-q)^{-\frac{1}{2}}) \otimes I]. \end{aligned} \tag{11}$$

Hollowood proved in [15] (in which this method was applied for the more general case of $Sl(n)$) that these definitions indeed yield new solutions of the YBE. The fusion procedure for the construction of higher R-matrices can be understood as the analogue for obtaining new irreducible representations of classical Lie algebras by considering spectral decompositions of tensor products of the fundamental representations.

$a_2^{(1)}$ ATFT contains three (mass degenerate) solitons, which all lie in the same multiplet V_1 transforming under the fundamental representation ρ_1. Analogously, their corresponding antisolitons transform under (ρ_2, V_2). Hollowood defined the following S-matrix, describing the scattering of these fundamental solitons and antisolitons,

$$S^{a,b}(\theta) \equiv X^{a,b}(x(\theta))\check{R}^{a,b}(x(\theta)) \qquad\qquad (a, b = 1, 2), \tag{12}$$

in which the connections between the deformation parameter q, the coupling constant β, the rapidity difference θ and the spectral parameter x are given by:

$$\begin{aligned} q &= -e^{-i\pi\lambda} \quad \text{, where } \lambda = \frac{4\pi}{\beta^2} - 1 \\ x &= e^{-i\pi\mu} \quad \text{, where } \mu = i\frac{3\lambda}{2\pi}\theta. \end{aligned} \tag{13}$$

(The form of the function λ has first been suggested by Arinshtein, Fateev and Zamolodchikov in [20].) The scalar factor $X^{a,b}$ in (12) ensures crossing symmetry and was given explicitly in terms of Gamma-functions in [15]. Hollowood was also able to give further evidence for this S-matrix by showing that the results from semiclassical calculations are in agreement with his S-matrix conjecture. Starting from this soliton S-matrix, in section 4 we will construct all S-matrix elements involving the scattering of bound states.

One readily discovers a similarity in the form of equations (10) and (11) with the form of the bootstrap equations (4). And indeed the fusion procedure to construct new solutions of the YBE is essentially the same procedure as constructing higher S-matrices from lower ones by applying the bootstrap equations. In the following section we will again make use of the fusion procedure by constructing bound states, which transform under irreducible representations in the tensor product of the two fundamental representations. In order to do this we will use a non-commutative algebra, as the Zamolodchikovs did for SG theory.

3 The Bound States

In this section we will construct explicit expressions of all possible bound states of two elementary solitons or antisolitons in $a_2^{(1)}$ ATFT (see also [15]).

As already mentioned above, $a_2^{(1)}$ ATFT contains three fundamental solitons and their corresponding antisolitons, which we will denote as A^j and $\overline{A}^j$ ($j = 1, 2, 3$) respectively. They are mass degenerate and their classical masses are $m_A = 2m\sin(\frac{\pi}{3})$. We will formally define a Zamolodchikov algebra in order to apply the same method the Zamolodchikovs used for SG theory. Due to the unconventional action of the charge conjugation operator defined in [15] it is necessary to redefine the in and out states in order to obtain explicit crossing symmetric expressions for the amplitudes of the transition, reflection and identical particle processes. In terms of the basis elements e_i ($i = 1, 2, 3$) of V_1 we redefine the in and out states as follows:

$$A^j(\theta_1)A^k(\theta_2) = e^{(j-k)\frac{1}{2}\lambda\theta_{12}} e_j \otimes e_k \qquad (j, k = 1, 2, 3). \tag{14}$$

$A^j(\theta_1)A^k(\theta_2)$ denotes two incoming solitons (of species j and k) if $\theta_1 > \theta_2$ and two outgoing solitons if $\theta_2 > \theta_1$. (This redefinition of states corresponds to a change from the homogeneous gradation to the principal gradation of the underlying charge algebra, as described in [21], and the same result could therefore equally be obtained by a suitable gauge transformation, as done in [17] for SG.) With this definition of the symbols $A^j(\theta)$ we can formally write the scattering processes of fundamental solitons in form of the following commutation relations:

$$\begin{aligned} A^j(\theta_1)A^j(\theta_2) &= S^I(\theta_{12})A^j(\theta_2)A^j(\theta_1) \\ A^j(\theta_1)A^k(\theta_2) &= S^T(\theta_{12})A^k(\theta_2)A^j(\theta_1) + S^{R(j,k)}(\theta_{12})A^j(\theta_2)A^k(\theta_1) \end{aligned} \tag{15}$$

in which S^T, $S^{R(j,k)}$ and S^I are scalar functions of one complex variable, which describe the transition, reflection and identical particle processes of fundamental solitons respectively. (The notation $S^{R(j,k)}$ indicates that the reflection amplitude, unlike in the SG case, depends on the species j, k of the two solitons.) In order to obtain these amplitudes from the quantum group S-matrix (12) we use the explicit expression (8) and let $S^{1,1}$ formally act on the in-states (14):

$$\begin{aligned} S^{1,1}(\theta_{12})[A^j(\theta_1)A^j(\theta_2)] &\equiv S^I(\theta_{12})A^j(\theta_2)A^j(\theta_1) \\ S^{1,1}(\theta_{12})[A^j(\theta_1)A^k(\theta_2)] &\equiv S^T(\theta_{12})A^k(\theta_2)A^j(\theta_1) + S^{R(j,k)}(\theta_{12})A^j(\theta_2)A^k(\theta_1). \end{aligned}$$

From this definition, using the relations (13), we obtain[2]:

$$\begin{aligned} S^I(\theta) &= 2i\sin(i\frac{3}{2}\lambda\theta + \lambda\pi)f_{0,0}(\mu) \\ S^T(\theta) &= 2i\sin(-i\frac{3}{2}\lambda\theta)f_{0,0}(\mu) \\ S^{R(j,k)}(\theta) &= 2i\sin(\lambda\pi)e^{\nu(j,k)\frac{1}{2}\lambda\theta}f_{0,0}(\mu) \end{aligned} \tag{16}$$

[2]These amplitudes coincide with those obtained by Nakatsu in [22] modulo some misprints.

$$\text{in which } \nu(j,k) \equiv \begin{cases} +1 & \text{if (j,k) = (1,2),(2,3) or (3,1)} \\ -1 & \text{if (j,k) = (2,1),(3,2) or (1,3).} \end{cases}$$

Our motivation for the change of notation for the scalar factor (our $f_{0,0}(\mu)$ is identical to $X^{1,1}(x(\theta))$ in Hollowood's notation) will become clearer below. The scalar factor $f_{0,0}(\mu)$ is given explicitly in section 4.

In order to obtain explicit expressions for the scattering amplitudes for processes involving antisolitons, in the case of $a_2^{(1)}$ it is sufficient to know the explicit expression (8) for $\check{R}^{1,1}$. Instead of using the fusion procedure (or equivalently the bootstrap equations) to obtain the higher S-matrices $S^{1,2}(\theta)$ and $S^{2,2}(\theta)$, we can get all the information about the scattering amplitudes by applying the crossing symmetry relation (equation (2)). Simply taking the crossed versions of the processes in (15), we obtain the following commutation relations:

$$\begin{aligned} A^j(\theta_1)\overline{A}^k(\theta_2) &= S^T(i\pi-\theta_{12})\overline{A}^k(\theta_2)A^j(\theta_1) \\ A^j(\theta_1)\overline{A}^j(\theta_2) &= S^I(i\pi-\theta_{12})\overline{A}^j(\theta_2)A^j(\theta_1) + S^{R(k,j)}(i\pi-\theta_{12})\overline{A}^k(\theta_2)A^k(\theta_1) + \\ &\quad + S^{R(l,j)}(i\pi-\theta_{12})\overline{A}^l(\theta_2)A^l(\theta_1) \end{aligned} \tag{17}$$

The relations for $\overline{A}-\overline{A}$ scattering processes are identical to relations (15).

Now we are able to formally construct bound states of fundamental solitons, using the same procedure as the Zamolodchikovs in [14]. The axioms of exact S-matrix theory state that bound states correspond to simple poles on the physical strip (i.e. $0 \le \mathrm{Im}(\theta) \le \pi$, $\mathrm{Re}(\theta) = 0$) in the direct channel of the S-matrix.

$\underline{S^I(\theta)}$ contains the following poles:

$$\theta = i\frac{2\pi}{3\lambda}p \qquad\qquad (\text{for } p = 1, 2, ... \le \frac{3}{2}\lambda), \tag{18}$$

which correspond to scalar bound states (bound states with zero topological charge, the so-called breathers) in the cross channel. Since the topological charge has to be conserved, these breathers must be bound states of a soliton-antisoliton pair of the same species. We have to add symbols B_p for breathers and $\overline{B}_p$ for their conjugate breathers to the algebra (15). Since these bound states are required to transform under irreducible representations, which appear in the spectral decomposition of the tensor product of ρ_1 with ρ_2, they must be defined as:

$$\begin{aligned} B_p(\frac{\theta_1+\theta_2}{2}) &= \lim_{\theta_2-\theta_1 \to -i\frac{2\pi}{3\lambda}p+i\pi} [\sum_{m=1}^{3} \alpha_m^{(p)} A^m(\theta_1)\overline{A}^m(\theta_2)] \\ \overline{B}_p(\frac{\theta_1+\theta_2}{2}) &= \lim_{\theta_2-\theta_1 \to -i\frac{2\pi}{3\lambda}p+i\pi} [\sum_{m=1}^{3} \alpha_m^{(p)} \overline{A}^m(\theta_1)A^m(\theta_2)] \end{aligned} \tag{19}$$

in which $p = 1, 2, ... \le \frac{3}{2}\lambda$, and

$$(\alpha_1^{(p)}, \alpha_2^{(p)}, \alpha_3^{(p)}) = \begin{cases} \eta(e^{i\frac{2\pi}{3}}, 1, e^{-i\frac{2\pi}{3}}) & \text{if p = 1,4,7,...} \\ \eta(e^{-i\frac{2\pi}{3}}, 1, e^{i\frac{2\pi}{3}}) & \text{if p = 2,5,8,...} \\ \eta(1,1,1) & \text{if p = 3,6,9,...} \end{cases}$$

(η is an arbitrary phase factor.)

The transition amplitude $\underline{S^T(\theta)}$ contains the following poles:

$$\theta = -i\frac{2\pi}{3\lambda}p + i\frac{2\pi}{3} \qquad\qquad (\text{for } p = 0, 1, ... \leq \lambda). \tag{20}$$

These poles correspond to bound states with non-zero topological charges, which we will call 'excited solitons'. We can define symbols $A_p^j(\theta)$ and $\overline{A}_p^j(\theta)$ for these excited solitons as:

$$\begin{aligned} A_p^j(\frac{\theta_1+\theta_2}{2}) &= \lim_{\theta_2-\theta_1 \to -i\frac{2\pi}{3\lambda}p+i\frac{2\pi}{3}} \eta[\gamma_1 \overline{A}^k(\theta_1)\overline{A}^l(\theta_2) + \gamma_2 \overline{A}^l(\theta_1)\overline{A}^k(\theta_2)] \\ \overline{A}_p^j(\frac{\theta_1+\theta_2}{2}) &= \lim_{\theta_2-\theta_1 \to -i\frac{2\pi}{3\lambda}p+i\frac{2\pi}{3}} \eta[\gamma_1 A^k(\theta_1)A^l(\theta_2) + \gamma_2 A^l(\theta_1)A^k(\theta_2)] \end{aligned} \tag{21}$$

for $p = 0, 1, ... \leq \lambda$ and $\{j, k, l\} = \{1, 2, 3\}$. We have to take $\gamma_1/\gamma_2 = (-1)^p e^{i\frac{\pi}{3}(\lambda-p)}$ (with an arbitrary phase factor η). The form of the parameters γ_j in the definition of A_p^j is determined by the requirement that the bound states transform under the quantum analogues of the irreducible representations, appearing in the tensor product of the fundamental representations ρ_1 or ρ_2 with themselves. Under which particular representation a given bound state transforms can be determined by the position of its corresponding pole in the spectral decomposition of the S-matrix. Considering the spectral decomposition (9), we see that the S-matrix poles, corresponding to excited solitons, appear in the factor in front of the projector s_2^-. This implies that the excited solitons A_p^j (antisolitons $\overline{A}_p^j$) must transform under the quantum group analogue of the fundamental representation ρ_1 (ρ_2). This requirement fixes the γ_j uniquely up to a phase factor η. To clarify this let us consider the example of $\overline{A}_p^1$: Using (21) and the redefinition (14) we see that with the given relation for γ_1 and γ_2 we have:

$$\overline{A}_p^1(\theta) \sim \frac{1}{q+q^{-1}}(e_2 \otimes e_3 - q e_3 \otimes e_2).$$

It can easily be proved that the expression on the right-hand side is indeed invariant under the action of the projector s_2^-, which was defined in section 2.3. The fact that $A_p^j(\theta)$ and $\overline{A}_p^j(\theta)$ transform under the same representations as the fundamental solitons $A^j(\theta)$ and $\overline{A}^j(\theta)$ further justifies the term 'excited solitons'. Note that for $p = 0$ these states are in fact identical to the elementary solitons and antisolitons.

Yet another important point to mention here is the fact that the poles corresponding to the excited solitons only appear in $S^T(\theta)$ and not in $S^I(\theta)$. This confirms the classical result obtained in [13] that there are no bound states of two solitons of the same species (i.e. <u>no</u> $A^j - A^j$ bound states) in the theory. There are no other poles apart from (18) and (20) in the soliton S-matrix, and so (19) and (21) give all possible bound states of two elementary solitons in $a_2^{(1)}$ ATFT.

4 Scattering of Bound States

The formal definitions of symbols A_p^j and B_p correspond to the calculation of the appropriate residues of the S-matrix elements and therefore enable us to calculate all bound state S-matrix elements solely by using the elementary algebra (15),(17) in the same way as done by the Zamolodchikovs for Sine-Gordon. This can be a rather elaborate procedure, for which we will only show one example in detail.

In order to obtain the scattering amplitude for the scattering of an elementary soliton A^j with a breather B_p we need to consider the triple product $A^j(\theta_1)\sum_{m=1}^{3}[\alpha_m^{(p)}A^m(\theta_2)\overline{A}^m(\theta_3)]$. Using the exchange relations (15) and (17) and taking the limit ($\theta_3-\theta_2 \to -i\frac{2\pi}{3\lambda}p+i\pi$) we obtain that the scattering process is purely a transition process, such that we have to add to the Zamolodchikov algebra the exchange relation:

$$A^j(\theta_1)B_p(\theta_2) = S_{AB_p}(\theta_{12})B_p(\theta_2)A^j(\theta_1).$$

The details of this calculation appear in Appendix B. There the transition amplitude for this process has been calculated to be

$$S_{AB_p}(\theta) = S^T(\theta - i\frac{\pi p}{3\lambda} + i\frac{\pi}{2})S^T(-\theta - i\frac{\pi p}{3\lambda} + i\frac{3\pi}{2})$$

By using the properties of the scalar factor $f(\mu)$ derived in Appendix C, we can simplify the expression for S_{AB_p} and end up with only a finite product of sine-functions:

$$S_{AB_p}(\theta) = \prod_{n=1}^{p} \frac{\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2n)+\frac{\pi}{4})\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2(n-1))+\frac{\pi}{4})}{\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2(n-1))+\frac{7}{12}\pi)\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2n)-\frac{\pi}{12})}. \tag{22}$$

By starting with the triple product $\sum_{m=1}^{3}[\alpha_m^{(p)}A^m(\theta_1)\overline{A}^m(\theta_2)]A^j(\theta_3)$ and taking the limit ($\theta_2-\theta_1 \to -i\frac{2\pi}{3\lambda}p+i\pi$), in the same way we obtain the amplitude for the transition process:

$$B_p(\theta_1)A^j(\theta_2) = S_{B_pA}(\theta_{12})A^j(\theta_2)B_p(\theta_1)$$

in which

$$S_{B_pA}(\theta) = \prod_{n=1}^{p} \frac{\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2n)-\frac{7}{12}\pi)\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2(n-1))+\frac{\pi}{12})}{\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2(n-1))-\frac{\pi}{4})\sin(\frac{\theta}{2i}+\frac{\pi}{6\lambda}(p-2n)-\frac{\pi}{4})}. \tag{23}$$

We would like to draw attention to the fact that $S_{AB_p}(\theta) \neq S_{B_pA}(\theta)$ which initially seems to be in violation of left-right symmetry. However, closer examination reveals that this phenomenon is closely related to the fact that the breathers of $a_2^{(1)}$-ATFT are not self-conjugate (i.e. $B_p \neq \overline{B}_p$), as in SG-theory. The breathers B_p and $\overline{B}_p$ are indeed two different kinds of particles despite the fact that both have the same mass and transform under the singlet representation. It is presently not clear to us whether they have different higher spin conserved local charges, like the mass degenerate particles in real ATFT (see [3]), or whether there are any non-local charges which serve to distinguish them uniquely.

Writing the breather states as bound states of fundamental solitons, *Figure 1* shows an $B_p - A^j$ scattering process on the left hand side and the same process with reversed ordering and reversed signs of the rapidities of the incoming particles on the right hand side. This illustrates the fact that the requirement of left-right symmetry implies $S_{AB_p}(\theta) = S_{\overline{B}_pA}(\theta)$ and not $S_{AB_p}(\theta) = S_{B_pA}(\theta)$. It can be seen easily that S_{AB_p} and S_{B_pA} are each separately crossing symmetric (i.e. $S_{AB_p}(i\pi - \theta) = S_{AB_p}(\theta)$) and therefore satisfy the required symmetry conditions:

$$\begin{aligned} S_{AB_p}(\theta) = S_{\overline{B}_pA}(\theta) = S_{\overline{A}\overline{B}_p}(\theta) = S_{B_p\overline{A}}(\theta); \\ S_{B_pA}(\theta) = S_{A\overline{B}_p}(\theta) = S_{\overline{B}_p\overline{A}}(\theta) = S_{\overline{A}B_p}(\theta). \end{aligned} \tag{24}$$

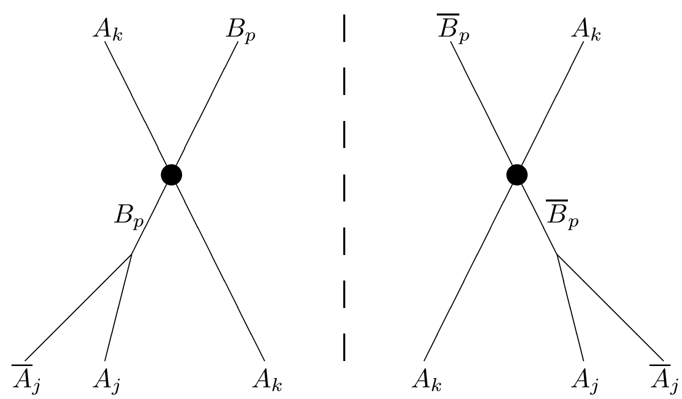

Figure 1: Left-right symmetry in S_{B_pA}

By further applying the same method we constructed all S-matrix elements for the scattering of all possible bound states. Without going into further detail of the calculations we will list the results here. We will use the following notations and abbreviations, some of which have been borrowed from references [3] and [15]:

$$\mu \equiv i\frac{3\lambda}{2\pi}\theta, \text{ and } \lambda \equiv \frac{4\pi}{\beta^2} - 1,$$

$$[x] \equiv sin(\frac{\theta}{2i} + \frac{\pi x}{6\lambda})$$

$$(x) \equiv \frac{sin(\frac{\theta}{2i} + \frac{\pi x}{6\lambda})}{sin(\frac{\theta}{2i} - \frac{\pi x}{6\lambda})} = \frac{[x]}{[-x]} \tag{25}$$

$$\text{and } \nu(j,k) \equiv \begin{cases} +1 & \text{if (j,k) = (1,2),(2,3) or (3,1)} \\ -1 & \text{if (j,k) = (2,1),(3,2) or (1,3).} \end{cases}$$

$a_2^{(1)}$ affine Toda field theory contains the following kinds of solitons, antisolitons and bound states:

a) *Fundamental solitons*: $A^j(\theta)$ $(j = 1, 2, 3)$
b) *Fundamental antisolitons*: $\overline{A}^j(\theta)$ $(j = 1, 2, 3)$
c) *Breathers and conjugate breathers ($A^j - \overline{A}^j$ bound states)*:
$B_p(\theta), \overline{B}_p(\theta)$ $(p = 1, 2, 3, ... \leq \frac{3}{2}\lambda)$
d) *Excited solitons and antisolitons ($A^l - A^k$ bound states)*:
$A_p^j(\theta), \overline{A}_p^j(\theta)$ $(j = 1, 2, 3;\ p = 0, 1, 2, ... \leq \lambda)$

Using these symbols all possible scattering processes can formally be written in terms of the following commutation relations:

a) soliton-soliton scattering:

$$\begin{aligned} A^j(\theta_1)A^j(\theta_2) &= S^I(\theta_{12})A^j(\theta_2)A^j(\theta_1) \\ A^j(\theta_1)A^k(\theta_2) &= S^T(\theta_{12})A^k(\theta_2)A^j(\theta_1) + S^{R(j,k)}(\theta_{12})A^j(\theta_2)A^k(\theta_1) \end{aligned}$$

b) soliton-breather scattering:

$$\begin{aligned} A^j(\theta_1)B_p(\theta_2) = S_{AB_p}(\theta_{12})B_p(\theta_2)A^j(\theta_1) \\ B_p(\theta_1)A^j(\theta_2) = S_{B_pA}(\theta_{12})A^j(\theta_2)B_p(\theta_1) \end{aligned}$$

c) breather-breather scattering:

$$B_r(\theta_1)B_p(\theta_2) = S_{B_rB_p}(\theta_{12})B_p(\theta_2)B_r(\theta_1)$$

d) excited soliton-breather scattering:

$$\begin{aligned} A_r^j(\theta_1)B_p(\theta_2) = S_{A_rB_p}(\theta_{12})B_p(\theta_2)A_r^j(\theta_1) \\ B_p(\theta_1)A_r^j(\theta_2) = S_{B_pA_r}(\theta_{12})A_r^j(\theta_2)B_p(\theta_1) \end{aligned}$$

e) excited soliton-excited soliton scattering:

$$\begin{aligned} A_p^j(\theta_1)A_r^j(\theta_2) &= S_{p,r}^I(\theta_{12})A_r^j(\theta_2)A_p^j(\theta_1) \\ A_p^j(\theta_1)A_r^k(\theta_2) &= S_{p,r}^T(\theta_{12})A_r^k(\theta_2)A_p^j(\theta_1) + S_{p,r}^{R(j,k)}(\theta_{12})A_r^j(\theta_2)A_p^k(\theta_1) \end{aligned}$$

Relations a) and b) can be regarded as special cases of d) and e).

The S-matrix elements are given by:

$$\begin{aligned} S^{I}(\theta) &= 2i\sin(i\frac{3}{2}\lambda\theta+\lambda\pi)f_{0,0}(\mu) \\ S^{T}(\theta) &= 2i\sin(-i\frac{3}{2}\lambda\theta)f_{0,0}(\mu) \\ S^{R(j,k)}(\theta) &= 2i\sin(\lambda\pi)e^{\nu(j,k)\frac{1}{2}\lambda\theta}f_{0,0}(\mu) \end{aligned} \tag{26}$$

$$\begin{aligned} S_{AB_p}(\theta) = \prod_{n=1}^{p} \frac{\left[p-2n+\frac{3}{2}\lambda\right]\left[p-2(n-1)+\frac{3}{2}\lambda\right]}{\left[p-2(n-1)+\frac{7}{2}\lambda\right]\left[p-2n-\frac{1}{2}\lambda\right]} \\ S_{B_pA}(\theta) = \prod_{n=1}^{p} \frac{\left[p-2n-\frac{7}{2}\lambda\right]\left[p-2(n-1)+\frac{1}{2}\lambda\right]}{\left[p-2(n-1)-\frac{3}{2}\lambda\right]\left[p-2n-\frac{3}{2}\lambda\right]} \end{aligned} \tag{27}$$

$$\begin{aligned} S_{B_rB_p}(\theta) &= \prod_{n=1}^{p}\Big(r+p-2n\Big)\Big(r+p-2(n-1)\Big) \\ &\quad \times\Big(p-r-2n-2\lambda\Big)\Big(p-r-2(n-1)+2\lambda\Big) \end{aligned} \tag{28}$$

$$\begin{aligned} S_{A_rB_p}(\theta) = \prod_{n=1}^{p} &\frac{\left[r+p-2n+\frac{3}{2}\lambda\right]\left[r+p-2(n-1)-\frac{1}{2}\lambda\right]}{\left[r+p-2(n-1)+\frac{7}{2}\lambda\right]\left[r+p-2n+\frac{7}{2}\lambda\right]} \\ &\times\frac{\left[p-r-2n+\frac{7}{2}\lambda\right]\left[p-r-2(n-1)+\frac{3}{2}\lambda\right]}{\left[p-r-2(n-1)-\frac{1}{2}\lambda\right]\left[p-r-2n-\frac{1}{2}\lambda\right]} \\ S_{B_pA_r}(\theta) = \prod_{n=1}^{p} &\frac{\left[r+p-2n+\frac{1}{2}\lambda\right]\left[r+p-2(n-1)+\frac{1}{2}\lambda\right]}{\left[r+p-2(n-1)+\frac{5}{2}\lambda\right]\left[r+p-2n-\frac{3}{2}\lambda\right]} \\ &\times\frac{\left[p-r-2n-\frac{7}{2}\lambda\right]\left[p-r-2(n-1)+\frac{5}{2}\lambda\right]}{\left[p-r-2(n-1)-\frac{3}{2}\lambda\right]\left[p-r-2n+\frac{1}{2}\lambda\right]} \end{aligned} \tag{29}$$

$$\begin{aligned} S^{I}_{p,r}(\theta) &= 2i\sin(i\frac{3}{2}\lambda\theta-\frac{\pi(p+r)}{2}+\lambda\pi)f_{p,r}(\mu) \\ S^{T}_{p,r}(\theta) &= 2i\sin(-i\frac{3}{2}\lambda\theta+\frac{\pi(p+r)}{2})f_{p,r}(\mu) \\ S^{R(j,k)}_{p,r}(\theta) &= 2i\sin(\lambda\pi)e^{\nu(j,k)(\frac{1}{2}\lambda\theta-i\frac{\pi(r+p)}{2})}f_{p,r}(\mu) \end{aligned} \tag{30}$$

For the sake of simplicity we only listed the relations for solitons and breathers, since all processes involving antisolitons or conjugate breathers can easily be obtained from these by taking their crossed version (i.e. $\theta \to i\pi - \theta$). For example the breather - conjugate breather scattering process is described by:

$$B_p(\theta_1)\overline{B}_r(\theta_2) = S_{B_p\overline{B}_r}(\theta_{12})\overline{B}_r(\theta_2)B_p(\theta_1)$$

in which

$$\begin{aligned} S_{B_p\overline{B}_r}(\theta) = S_{B_rB_p}(i\pi - \theta) \;\; &= \;\; \prod_{n=1}^{p}\Big(p - r - 2n + 3\lambda\Big)\Big(p - r - 2(n-1) + 3\lambda\Big) \\ &\quad \times\Big(r + p - 2(n-1) + 5\lambda\Big)\Big(r + p - 2n + \lambda\Big) \end{aligned} \tag{31}$$

It can easily be proven that these breather-breather scattering amplitudes satisfy the obvious symmetry conditions:

$$\begin{aligned} S_{B_rB_p}(\theta) = S_{B_pB_r}(\theta) = S_{\overline{B}_r\overline{B}_p}(\theta) = S_{\overline{B}_p\overline{B}_r}(\theta); \\ S_{\overline{B}_pB_r}(\theta) = S_{B_r\overline{B}_p}(\theta) = S_{\overline{B}_rB_p}(\theta) = S_{B_p\overline{B}_r}(\theta). \end{aligned}$$

All the scalar factors in the expressions (26) and (30) can be expressed in terms of the fundamental function $f(\mu)$ given by Hollowood:

$$\begin{aligned} f(\mu) = \frac{1}{2i\sin(\pi(\mu+\lambda))}\prod_{j=1}^{\infty}\frac{\Gamma[1+\mu+(3j-3)\lambda]\Gamma[\mu+3j\lambda]}{\Gamma[1-\mu+(3j-3)\lambda]\Gamma[-\mu+3j\lambda]} \\ \times\frac{\Gamma[-\mu+(3j-2)\lambda]\Gamma[1-\mu+(3j-1)\lambda]}{\Gamma[\mu+(3j-2)\lambda]\Gamma[1+\mu+(3j-1)\lambda]}. \end{aligned} \tag{32}$$

In Appendix C we show the calculations which led to the following expression of $f_{p,r}$:

$$\begin{aligned} f_{p,r}(\mu) = \quad & \Big(r+p\Big)\Big(-r-p-2\lambda\Big)\Big(p-r+2\lambda\Big)\Big(r-p+2\lambda\Big) \\ & \times\textstyle\prod_{n=1}^{r}\frac{[r+p-2(n-1)+2\lambda][r+p-2n]}{[r+p-2n-2\lambda][r+p-2(n-1)-2\lambda]} \\ & \times\textstyle\prod_{n=1}^{p}\frac{[p+r-2n][p+r-2(n-1)]}{[p+r-2(n-1)-2\lambda][p+r-2n+2\lambda]}\frac{[p-r-2(n-1)+2\lambda][p-r-2n+2\lambda]}{[p-r-2(n-1)][p-r-2n-2\lambda]} \\ & \times f(\mu-\tfrac{p+r}{2}) \end{aligned} \tag{33}$$

$$(\text{for } p,r = 0,1,2,... \leq \lambda).$$

Before we discuss the analytic structure of these S-matrix elements in the following section, we should mention one interesting result which emerges here: in accordance with the results of SG theory (see [14]) and Smirnov's work on $a_2^{(2)}$ ATFT (see [23]) it seems natural to identify the two lowest breather states (B_1 and $\overline{B}_1$) with the two fundamental

quantum particles, which are the only states present in real ATFT. The two fundamental particles of real $a_2^{(1)}$ ATFT are mass degenerate and are conjugate to each other.

Hollowood calculated the first quantum mass corrections for the solitons of $a_n^{(1)}$-ATFT in [24]. Using the conjectured soliton S-matrix he was able to show that the masses of the quantum particles (calculated up to one-loop order) are indeed identical to the masses of the lowest breather states including these first quantum corrections. We are now in a position to further justify this identification by comparing the breather-breather S-matrix elements with the real ATFT S-matrix given by Braden et al. in [3]. If we consider the two lowest breather states B_1 and $\overline{B}_1$, we get from formula (28):

$$\begin{aligned} S_{B_1 B_1}(\theta) &= \Big(2\Big)\Big(2\lambda\Big)\Big(-2-2\lambda\Big) \\ S_{\overline{B}_1 B_1}(\theta) &= \Big(-2+3\lambda\Big)\Big(3\lambda\Big)\Big(2+5\lambda\Big)\Big(\lambda\Big). \end{aligned}$$

If we change the imaginary coupling constant from $i\beta$ to β (β real) in these two functions, then we indeed obtain the expression of the conjectured exact S-matrix for real $a_2^{(1)}$ ATFT (see [3]). Thus the lowest breather states B_1 and $\overline{B}_1$ in the complex theory can indeed be identified with the two fundamental quantum particles 1 and 2. This fact also gives further support to the quantum group S-matrix conjecture by Hollowood. However, a full understanding of the origin and the implications of this duality between quantum particles and bound states of solitons is still missing, and further work is needed.

5 Pole Structure

In this section we attempt to explain the poles in the S-matrix elements obtained in the previous section. (Here we shall take λ to be sufficiently large to admit several breathers and excited solitons into the spectrum. We also assume that 3λ is not an integer, since in this case different poles frequently coincide and it becomes rather more complicated to explain the entire pole structure. The case in which 3λ or λ are integers requires separate examination.)

We will conjecture a complete set of three-point couplings of the theory and try to explain most other poles either by simple tree level and box diagrams or by a generalized Coleman-Thun mechanism. Which poles correspond to fusion processes can be determined by using the bootstrap principle. The fusions of two elementary solitons into bound states are illustrated in *Figure 2a* and *2b*. Using the mass formula (3) we can calculate the masses of breathers and excited solitons from their corresponding S-matrix pole and obtain (see [15]):

$$\begin{aligned} m_{B_p} &= 2m_A \sin(\frac{\pi p}{3\lambda}) \\ m_{A_p} &= 2m_A \cos(\frac{\pi}{3} - \frac{\pi p}{3\lambda}). \end{aligned} \tag{34}$$

In the same way formula (3) can be used to find other three point couplings by looking at the poles in higher S-matrix elements. So for instance, the process in *Fig.2d* can exists,

since the element $S^I_{r,p}(i\pi-\theta)$ contains a pole $\theta=-\frac{i\pi}{3\lambda}(p-r)+i\pi$ and it is easily checked that:

$$m^2_{B_{p-r}}=m^2_{A_p}+m^2_{A_r}+2m_{A_p}m_{A_r}cos(-\frac{\pi}{3\lambda}(p-r)+\pi).$$

The existence of the process *Fig.2d* can then additionally be checked by applying the bootstrap equations (4). Using the same method all other three-point vertices in *Figure 2* are found and the vertices *Fig.2a - 2f* are conjectured to be a complete list of all possible three point vertices in the theory, that is, all other possible fusion processes are simply crossed or conjugated versions of these six vertices. These three-point couplings will be used below to explain other poles through higher order on-shell diagrams. (In these and all following diagrams time is meant to be moving 'upwards'.)

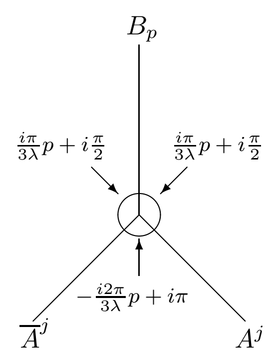

Figure 2a

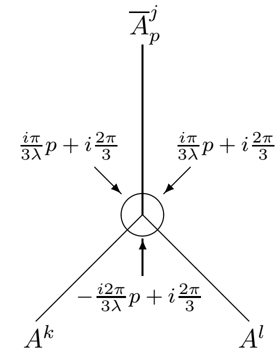

Figure 2b

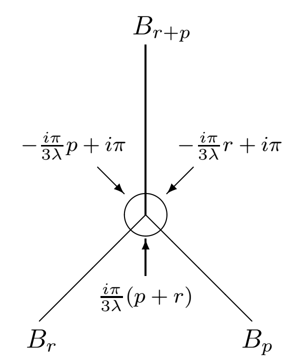

Figure 2c

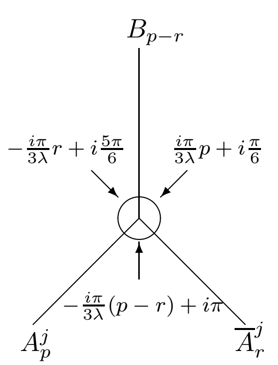

Figure 2d

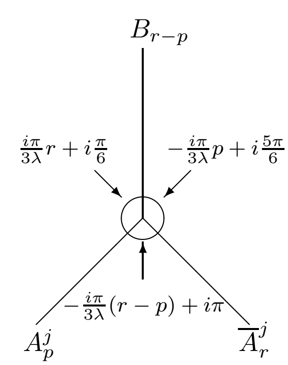

Figure 2e

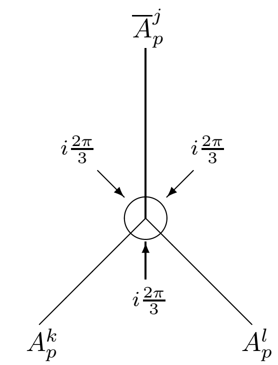

Figure 2f

The (imaginary) angles in the diagrams represent the rapidity difference of the particles. The sum of the values of the rapidities in each diagram is $2\pi i$.

As mentioned above, all poles in the S-matrix elements describing the scattering of elementary solitons were already explained by fusion into bound states. In order to explain the mechanisms responsible for the occurence of poles in other S-matrix elements, we will discuss the pole structure of the elements S_{B_pA} and S_{AB_p} in detail. For poles in all other S-matrix elements we will restrict ourself to giving the relevant diagrams in Appendix B.

i) S_{B_pA}

S_{B_pA} contains the following poles:

$$\theta_n = -\frac{i\pi}{3\lambda}(p-2n) + i\frac{\pi}{2} \qquad \begin{array}{l}\text{(simple for } n=0,p)\\ \text{(double for } n=1,2,...,p-1)\end{array} \qquad (35)$$

The simple pole θ_p corresponds to the fusion process $B_p + A^j \rightarrow A^j$, and is therefore explained by the tree-level diagram *Fig.3a*. Since $\theta_o = i\pi - \theta_p$, the other simple pole is then simply explained by the crossed version of this tree-level process.

To explain double poles in a two dimensional S-matrix one has to consider box diagrams like *Fig.3b* or crossed box diagrams like *Fig.4*. The Cutkosky rules in two dimensions state that in an on-shell diagram every loop contributes a two-dimensional integral whereas every internal line contributes a delta-function [2]. Thus, if diagrams like *Fig.3b* or *Fig.4* with all internal particles on-shell exist, then they correspond to double poles in the direct channel of the S-matrix. This phenomenon was first observed by Coleman and Thun in [25] for the Sine-Gordon theory, and is therefore called the Coleman-Thun mechanism.

Since the (purely imaginary) rapidity differences correspond to (real) angles in the diagrams, we can use elementary Euclidean geometry to study higher order processes, such as *Fig.3b*, *Fig.4* or any of the diagrams in Appendix D. All angles in diagram *Fig.3b* are fixed by the three point vertices (*Fig.2a-f*). It is therefore straightforward to calculate the rapidity difference of the incoming particles. We obtain that the process pictured in *Fig.3b* can occur if the incoming particles B_p and A^j have a rapidity difference of $\theta = -\frac{i\pi}{3\lambda}(p-2s) + i\frac{\pi}{2}$. Since s can take values $1 \leq s \leq p-1$, these values of the rapidity difference are exactly the double poles in (35). Thus all poles in S_{B_pA} are explained by the two scattering processes in *Fig.3*.

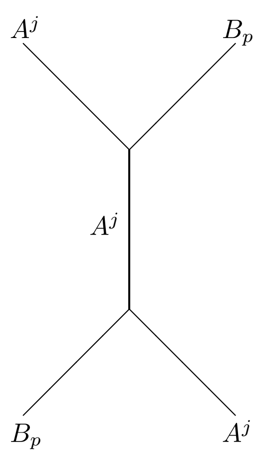

Figure 3a

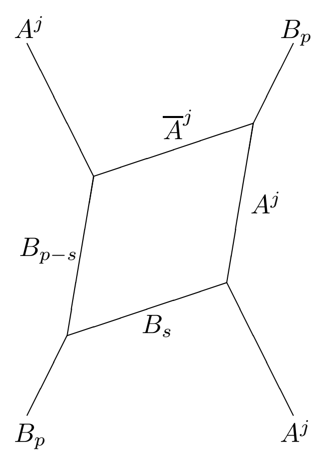

Figure 3b

ii) S_{AB_p}

S_{AB_p} contains no double poles in general, but the following simple poles:

$$\begin{aligned} \theta_n^{(a)} &= -\frac{i\pi}{3\lambda}(p-2n) + i\frac{\pi}{6} \\ \theta_n^{(b)} &= -\frac{i\pi}{3\lambda}(p-2(n-1)) + i\frac{5\pi}{6} \qquad (\text{for } n = 1,2,...,p) \end{aligned} \tag{36}$$

(it should be noted that for certain values of n some of these poles might not lie on the physical strip, and can therefore be ignored).

The simple poles $\theta_p^{(a)}$ and $\theta_1^{(b)}$ correspond to the fusion process $A^j + B_p \rightarrow A_p^j$ (see *Fig.2d* for $r = 0$) in the direct and crossed channel respectively. In order to explain all other poles we encounter a new phenomenon, the so-called generalized Coleman-Thun mechanism, explained in detail in [26]. This mechanism corresponds to diagrams like *Fig.4*, which initially would be expected to lead to double poles, as explained above. However, we need to consider the transition process of the two particles in the center of the diagram (indicated by a black circle in the diagram). If it happens that the S-matrix element of this process exhibits a simple zero at exactly the right rapidity difference, then this zero reduces the expected double pole, and a diagram like *Fig.4* leads to a simple pole in the direct channel of $A^j - B_p$ scattering. This mechanism occurs in the case of S_{AB_p}.

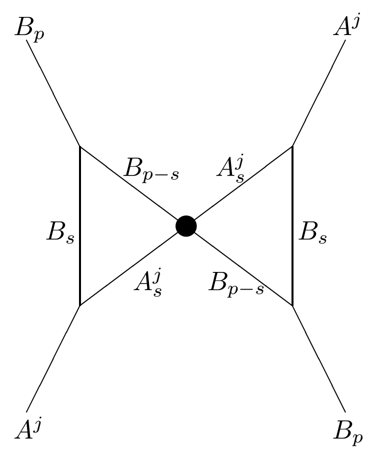

Figure 4

Using diagrams *Fig.2a-f* and elementary geometrical calculations we can compute all angles in diagram *Fig.4* and obtain for the rapidity difference of the incoming particles A^j and B_p:

$$\theta = -\frac{i\pi}{3\lambda}(2s-p) + i\frac{5\pi}{6} \qquad (= \theta_{p+1-s}^{(b)})$$

in which s can take values $1 \leq s \leq p-1$. The rapidity difference for the incoming internal particles A_s^j and B_{p-s} is $\theta = \frac{i\pi}{3\lambda}p + i\frac{5\pi}{6}$, which is a simple zero in $S_{A_s B_{p-s}}(\theta)$. Thus it emerges that the transition process in the center of this diagram indeed occurs at a

rapidity difference where the corresponding transition amplitude exhibits a simple zero, which reduces the expected double poles to simple poles. Thus we reach the result that all remaining simple poles $\theta_n^{(b)}$ $(n = 2, 3, ..., p)$ in S_{AB_p} are explained by this generalized Coleman-Thun mechanism. Since S_{AB_p} is itself crossing symmetric, the poles $\theta_n^{(a)}$ are just cross channel poles of $\theta_{p-n}^{(b)}$ and therefore correspond to the same process in the crossed channel.

In the following we will restrict ourself to giving the poles of the S-matrix elements and refer to the corresponding diagrams in Appendix D.

(iii) $S_{B_rB_p}$

For the sake of simplicity let us assume here $r \geq p$. $S_{B_rB_p}$ contains the poles:

$$\begin{array}{rclll}
\theta_n^{(a)} & = & \frac{i\pi}{3\lambda}(p+r-2n) & \text{(simple for } n = 0, p) & \\
 & & & \text{(double for } n = 1, 2, ..., p-1) & \\
\theta_n^{(b)} & = & -\frac{i\pi}{3\lambda}(r-p+2n) + i\frac{4\pi}{3} & \text{(simple for } n = 1, 2, ..., p) & \\
\theta_n^{(c)} & = & -\frac{i\pi}{3\lambda}(r-p+2(n-1)) + i\frac{2\pi}{3} & \text{(simple for } n = 1, 2, ..., p). & (37)
\end{array}$$

The two simple poles $\theta_0^{(a)}$ and $\theta_p^{(a)}$ are due to the fusion processes $B_r + B_p \rightarrow B_{r+p}$ in the direct channel and $B_p + \overline{B}_r \rightarrow \overline{B}_{r-p}$ in the cross channel (*Fig.2c*), whereas the double poles $\theta_n^{(a)}$ are explained by the crossed box diagram in Appendix D (*Fig.5a*). Unlike in the former case, here the transition amplitude does not have a zero, so that this diagram leads to the expected double pole in the S-matrix.

There are neither tree level nor box diagrams which could explain the remaining poles in $S_{B_rB_p}$, but there exist higher order diagrams. We will give one example of a third order process. In the diagram, depicted in Appendix D *Fig.5b*, the rapidity difference of the incoming particles is equal to $\theta_n^{(b)}$ (for $s = p - n$). This diagram contains 5 loops and 13 internal lines, and should therefore lead to a triple pole. However, a careful study of all internal angles reveals that two of the three transition processes in the diagram occur at a rapidity difference, where the corresponding transition amplitudes exhibit a simple zero. These zeros reduce the expected triple poles to the simple poles $\theta_n^{(b)}$ $(n = 1, 2, ...p)$. We were not able to find a second or third order diagram which could explain the remaining string of poles $\theta_n^{(c)}$. We expect these poles to correspond to fourth or even higher order diagrams.

(iv) $S_{A_rB_p}$

This element contains four simple poles and two strings of double poles:

$$\begin{array}{rclll}
\theta_n^{(a)} & = & -\frac{i\pi}{3\lambda}(r+p-2n) + i\frac{5\pi}{6} & \text{(simple for } n = 0, p) & \\
 & & & \text{(double for } n = 1, 2, ..., p-1) & \\
\theta_n^{(b)} & = & -\frac{i\pi}{3\lambda}(p-r-2n) + i\frac{\pi}{6} & \text{(simple for } n = 0, p) & \\
 & & & \text{(double for } n = 1, 2, ..., p-1) & (38)
\end{array}$$

The four simple poles can be explained by tree level processes. $\theta_p^{(b)}$ and $\theta_0^{(a)}$ correspond to the fusion process $A_r^j + B_p \rightarrow A_{r+p}^j$ (*Fig.2d*) in the direct and crossed channel. $\theta_p^{(a)}$ and

$\theta_0^{(b)}$ correspond to the fusion process $A_r^j + B_p \to A_{r-p}^j$ (*Fig.2e*). The double poles are due to the process illustrated by the box diagram *Fig.6* in the direct and crossed channel.

(iv) $S_{B_pA_r}$

This element contains four strings of simple poles:

$$\begin{aligned}
\theta_n^{(a)} &= -\frac{i\pi}{3\lambda}(r+p-2(n-1)) + i\frac{7\pi}{6} \\
\theta_n^{(b)} &= -\frac{i\pi}{3\lambda}(p-r-2n) - i\frac{\pi}{6} \\
\theta_n^{(c)} &= -\frac{i\pi}{3\lambda}(r+p-2n) + i\frac{\pi}{2} \\
\theta_n^{(d)} &= -\frac{i\pi}{3\lambda}(p-r-2(n-1)) + i\frac{\pi}{2} \quad \text{(for } n = 1, 2, ..., p).
\end{aligned}$$

Since $\theta^{(a)}$ and $\theta^{(b)}$, as well as $\theta^{(c)}$ and $\theta^{(d)}$, are crossed channel poles to each other it is sufficient to find corresponding diagrams to only two of these strings of poles. $\theta_1^{(d)}$ (and therefore also $\theta_p^{(c)}$) can be explained by the process in *Fig.7a*. The diagram *Fig.7b* explains the poles $\theta_n^{(a)}$ and $\theta_n^{(b)}$ ($n = 1, 2, ..., p$), whereas the remaining poles $\theta_n^{(c)}$ and $\theta_{p-n}^{(d)}$ (for $n = 2, 3, ..., p$) are due to the third order process in *Fig.7c*, which displays (similar to *Fig.5b*) a 'double' generalized Coleman-Thun mechanism.

(iv) $S_{A_pA_r}$

The S-matrix elements for the scattering of excited solitons exhibit a large number of simple and multiple poles. We were not able to explain all poles in a general way as done for the other S-matrix elements. However, we were able to explain the pole structure for some simple explicit examples through low order diagrams and the generalized Coleman-Thun mechanism by using the above list of three point couplings. We expect that all simple and higher order poles in these S-matrix elements can be explained in this way. However, in many cases the relevant diagrams can become extremely complicated and may therefore prove more elusive.

We do not have a direct way of proving that the list of three-point couplings in *Figure 2a-f*, is complete and that there are no other fusion processes possible. However, the fact that we were able to explain almost all other poles in the S-matrix through higher order diagrams using only those six three-point vertices provides strong evidence for our conjecture that the list of three point couplings is indeed complete.

6 Discussion

Apart from SG-theory ($a_1^{(1)}$ ATFT), the complete spectra of solitons and bound states in any other complex ATFT have been heretofore unknown. In this paper we have attempted to solve this problem for the subsequent (and more complicated) $a_2^{(1)}$ affine Toda field theory. We have constructed bound states of elementary solitons and conjectured that these

bound states together with the solitons and their antisolitons form the complete spectrum of the theory. We have calculated explicit S-matrix elements for the scattering of solitons and bound states and have been able to explain their pole structure by using a surprisingly small number of three point couplings. The S-matrix elements for the lowest breather states have been shown to coincide with the S-matrix for the fundamental quantum particles, which provides a further justification for this S-matrix conjecture.

The discussion in this paper has been restricted to the case where 3λ is not an integer. This case should be considered separately.

It is necessary to further consider the question of unitarity. Although the theory is non-unitary in general Hollowood suggested in [15] that the S-matrix seems to describe a unitary field theory in the strong coupling limit (i.e. if λ is sufficiently small). One could also ask whether certain restrictions of the theory, like RSOS restrictions or the restriction to integer values of λ, would lead to unitary theories.

Another question unmentioned in this paper relates to the nature of the possible connection of the residues of poles and the parity of bound states with the problem of unitarity, first discussed in [27]. The task of calculating the pole residues of our S-matrix elements seems rather difficult, but it would be interesting to examine in what way the pole residues change sign with the change of λ (an interesting connection of the change of residue signs with the generalized Coleman-Thun mechanism has been discovered in [26]).

The question arises as to whether our method is equally applicable to the more general case of $a_n^{(1)}$ ATFT's. Although more elaborate, this application seems to be possible in principle. However, an additional problem surfaces for $n \geq 3$, namely the problem of unfilled weights of the fundamental representations. This problem has been examined in detail by McGhee in [28] and it emerged that for the $a_n^{(1)}$ series of ATFT only in the case $n = 1$ and $n = 2$ the topological charges of the soliton solutions fill out the weight lattice of the fundamental representations. It is presently not clear how in the quantum case this could affect the conjectured S-matrix.

Another possible area for further research would be the search for a more rigorous definition of the generators and commutation relations of the Zamolodchikov algebra (15), as was done in terms of vertex operators in [29].

The question also remains as to how to construct a consistent soliton S-matrix for theories based on other algebras and in particular on non-simply laced or twisted algebras. In two recent publications ([30], [31]) the first quantum mass corrections of affine Toda solitons were calculated by using semiclassical techniques. It was found that for many theories the soliton masses do not renormalise according to the masses of the quantum particles. In these cases the approach of defining an S-matrix in terms of the basic R-matrix, seems not to be applicable in the same way as for the $a_n^{(1)}$ theories. A possible solution to this problem may lie in using R-matrices with a different gradation (see [32]).

Acknowledgement

I would like to thank N.J. MacKay for introducing me to this subject and for many discussions during the course of this work.

APPENDIX A: Sine-Gordon S-Matrix

In this appendix the complete S-matrix for the Sine-Gordon model, as derived in [14], is given. We use the following notations, which helps to illustrate the connection with our S-matrix elements for $a_2^{(1)}$-ATFT.

$$
\begin{aligned}
[x]_S &\equiv sin(\frac{\theta}{2i}+\frac{\pi x}{4\lambda}) \\
(x)_S &\equiv \frac{[x]_S}{[-x]_S}.
\end{aligned}
$$

For convenience we also use the abbreviations $\tilde{\mu} \equiv i\frac{\lambda}{\pi}\theta$ and $\lambda \equiv \frac{4\pi}{\beta^2}-1$.
The commutation relations for the complete Zamolodchikov algebra are the following:

$$
\begin{aligned}
A(\theta_1)A(\theta_2) &= \tilde{S}^I(\theta_{12})A(\theta_2)A(\theta_1) \\
A(\theta_1)\overline{A}(\theta_2) &= \tilde{S}^T(\theta_{12})\overline{A}(\theta_2)A(\theta_1)+\tilde{S}^R(\theta_{12})A(\theta_2)\overline{A}(\theta_1) \\
A(\theta_1)B_p(\theta_2) &= \tilde{S}_p(\theta_{12})B_p(\theta_2)A(\theta_1) \\
B_r(\theta_1)B_p(\theta_2) &= \tilde{S}_{r,p}(\theta_{12})B_p(\theta_2)B_r(\theta_1)
\end{aligned}
$$

and the S-matrix elements are given by:

$$
\begin{aligned}
\tilde{S}^I(\theta) &= 2i\sin(i\lambda\theta+\lambda\pi)f_{SG}(\tilde{\mu}) \\
\tilde{S}^T(\theta) &= 2i\sin(-i\lambda\theta)f_{SG}(\tilde{\mu}) \\
\tilde{S}^R(\theta) &= 2i\sin(\lambda\pi)f_{SG}(\tilde{\mu}) \\
\tilde{S}_p(\theta) &= \prod_{n=1}^{p}\frac{[2n-p-3\lambda]_S\,[p-2n+\lambda]_S}{[p-2n-\lambda]_S\,[2n-p-\lambda]_S} \\
\tilde{S}_{r,p}(\theta) &= \prod_{n=1}^{p}\Big(r+p-2n\Big)_S\Big(r+p-2(n-1)\Big)_S\Big(p-r-2n-2\lambda\Big)_S\Big(p-r-2(n-1)+2\lambda\Big)_S.
\end{aligned}
$$

The scalar factor $f_{SG}(\tilde{\mu})$ ensures crossing symmetry and can be written in the following form:

$$
\begin{aligned}
f_{SG}(\tilde{\mu}) = \frac{1}{2i\sin(\pi(\tilde{\mu}+\lambda))}\prod_{j=1}^{\infty}&\frac{\Gamma[1+\tilde{\mu}+(2j-2)\lambda]\Gamma[\tilde{\mu}+2j\lambda]}{\Gamma[1-\tilde{\mu}+(2j-2)\lambda]\Gamma[-\tilde{\mu}+2j\lambda]} \\
&\times\frac{\Gamma[-\tilde{\mu}+(2j-1)\lambda]\Gamma[1-\tilde{\mu}+(2j-1)\lambda]}{\Gamma[\tilde{\mu}+(2j-1)\lambda]\Gamma[1+\tilde{\mu}+(2j-1)\lambda]}.
\end{aligned}
$$

APPENDIX B: Calculation of S_{AB_p}

In this appendix we show the calculations used to obtain the commutation relation of $A^j(\theta)$ $(j = 1, 2, 3)$ with $B_p(\theta)$. For convenience let us define the breather pole:

$$\Theta_p \equiv -i\frac{2\pi}{3\lambda}p + i\pi$$

Regarding the formal definition (19) of the symbols B_p, we need to consider the triple product $A^j(\theta_1)\sum_{m=1}^{3}[\alpha_m^{(p)}A^m(\theta_2)\overline{A}^m(\theta_3)]$. Using the commutation relations for elementary solitons and their antisolitons (15) and (17), we obtain the following expression.

$$\begin{aligned}
&A^j(\theta_1)\sum_{m=1}^{3}[\alpha_m^{(p)}A^m(\theta_2)\overline{A}^m(\theta_3)] = \\
&\quad \alpha_j^{(p)}\Big\{S^I(\theta_{12})S^I(i\pi-\theta_{13}) + \frac{\alpha_k^{(p)}}{\alpha_j^{(p)}}S^{R(j,k)}(\theta_{12})S^{R(j,k)}(i\pi-\theta_{13}) \\
&\qquad\qquad + \frac{\alpha_l^{(p)}}{\alpha_j^{(p)}}S^{R(j,l)}(\theta_{12})S^{R(j,l)}(i\pi-\theta_{13})\Big\}_{(1)} A^j(\theta_2)\overline{A}^j(\theta_3)A^j(\theta_1) \\
&\quad +\alpha_k^{(p)}\Big\{S^T(\theta_{12})S^T(i\pi-\theta_{13})\Big\}_{(2)} A^k(\theta_2)\overline{A}^k(\theta_3)A^j(\theta_1) \\
&\quad +\alpha_l^{(p)}\Big\{S^T(\theta_{12})S^T(i\pi-\theta_{13})\Big\}_{(3)} A^l(\theta_2)\overline{A}^l(\theta_3)A^j(\theta_1) \\
&\quad +\Big\{\alpha_j^{(p)}S^I(\theta_{12})S^{R(k,j)}(i\pi-\theta_{13}) + \alpha_k^{(p)}S^{R(j,k)}(\theta_{12})S^I(i\pi-\theta_{13}) \\
&\qquad\qquad +\alpha_l^{(p)}S^{R(j,l)}(\theta_{12})S^{R(k,l)}(i\pi-\theta_{13})\Big\}_{(4)} A^j(\theta_2)\overline{A}^k(\theta_3)A^k(\theta_1) \\
&\quad +\Big\{\alpha_j^{(p)}S^I(\theta_{12})S^{R(l,j)}(i\pi-\theta_{13}) + \alpha_l^{(p)}S^{R(j,l)}(\theta_{12})S^I(i\pi-\theta_{13}) \\
&\qquad\qquad +\alpha_k^{(p)}S^{R(j,k)}(\theta_{12})S^{R(l,k)}(i\pi-\theta_{13})\Big\}_{(5)} A^j(\theta_2)\overline{A}^l(\theta_3)A^l(\theta_1).
\end{aligned} \tag{39}$$

In order to obtain the required scattering amplitude we have to take the limit $(\theta_3-\theta_2 \to \Theta_p)$ on both sides of this expression. By taking this limit we obtain the following and rather simple identities for the factors on the right hand side of (39):

$$\lim_{\theta_3-\theta_2\to\Theta_p}\Big\{\ \Big\}_{(1)} = \lim_{\theta_3-\theta_2\to\Theta_p}\Big\{\ \Big\}_{(2)} = \lim_{\theta_3-\theta_2\to\Theta_p}\Big\{\ \Big\}_{(3)};$$

$$\text{and}\qquad \lim_{\theta_3-\theta_2\to\Theta_p}\Big\{\ \Big\}_{(4)} = \lim_{\theta_3-\theta_2\to\Theta_p}\Big\{\ \Big\}_{(5)} = 0.$$

Since the last two terms vanish, we see that the only scattering process possible is a transition of the two incoming particles. Hence we introduce the commutation relation:

$$A^j(\theta_1)B_p(\frac{\theta_2+\theta_3}{2}) = S_{AB_p}(\theta)B_p(\frac{\theta_2+\theta_3}{2})A^j(\theta_1)$$

in which $\theta = \theta_1 - \frac{\theta_2+\theta_3}{2}$. According to expression (39) the scalar function $S_{AB_p}(\theta)$ must be

$$S_{AB_p}(\theta) \equiv \lim \Big\{\Big\}_{(2)} = \lim_{\theta_3-\theta_2\to\Theta_p} S^T(\theta_{12}) S^T(i\pi - \theta_{13})$$

or in terms of θ:

$$\begin{aligned} S_{AB_p}(\theta) &= S^T(\theta + \frac{1}{2}\Theta_p) S^T(i\pi - \theta + \frac{1}{2}\Theta_p) \\ &= (-4)\sin(-i\frac{3}{2}\lambda\theta - \frac{\pi p}{2} + \frac{3}{4}\lambda\pi)\sin(i\frac{3}{2}\lambda\theta - \frac{\pi p}{2} + \frac{9}{4}\lambda\theta) \\ &\quad \times f_{0,0}(\mu + \frac{p}{2} - \frac{3}{4}\lambda) f_{0,0}(-\mu + \frac{p}{2} - \frac{9}{4}\lambda). \end{aligned} \tag{40}$$

The calculation of the other S-matrix elements, listed in section 4, proceeds on similar lines.

APPENDIX C: Some properties of the scalar factor

In order to simplify expressions such as (40) we need to state some properties of the scalar factor $f_{0,0}(\mu)$, which has been computed by Hollowood in [15] to be $f_{0,0}(\mu) = (2\lambda) f(\mu)$ and

$$\begin{aligned} f(\mu) = \frac{1}{2i\sin(\pi(\mu+\lambda))} \prod_{j=1}^{\infty} & \frac{\Gamma[1+\mu+(3j-3)\lambda]\Gamma[\mu+3j\lambda]}{\Gamma[1-\mu+(3j-3)\lambda]\Gamma[-\mu+3j\lambda]} \\ & \times \frac{\Gamma[-\mu+(3j-2)\lambda]\Gamma[1-\mu+(3j-1)\lambda]}{\Gamma[\mu+(3j-2)\lambda]\Gamma[1+\mu+(3j-1)\lambda]}. \end{aligned} \tag{41}$$

By using the elementary property of the Γ-function: $\Gamma(z+1) = z\Gamma(z)$ (for $z \neq 0, -1, -2, ...$), and the product expansion of the sine-function:

$$\sin(x) = x \prod_{n=1}^{\infty} (1 - \frac{x^2}{k^2\pi^2}) \tag{42}$$

we can derive the following properties of $f(\mu)$

$$\begin{aligned} f(\mu - p) &= (-1)^p \prod_{n=1}^{p} \frac{[2(n-1)+2\lambda]\,[2n-2\lambda]}{[2(n-1)]\,[2n]} f(\mu) \\ f(\mu)f(-\mu-3\lambda) &= \frac{1}{4\sin(\pi\mu)\sin(\pi(\mu+3\lambda))} \\ f(\mu)f(\mu-\lambda) &= \frac{-\sin(\pi(\mu-3\lambda))}{2i\sin(\pi(\mu+\lambda))\sin(\pi(\mu-2\lambda))} f(-\mu+2\lambda) \end{aligned} \tag{43}$$

in which p is a positive integer and the notation $[x]$ has been used as defined in section 4. Additionally we trivially have

$$f(-\mu) = \frac{1}{-4\sin^2(\pi(\mu+\lambda))}(f(\mu))^{-1}. \tag{44}$$

It should be mentioned that we assumed in these calculations that λ is not an integer. This case has to be considered separately, since there can appear additional singularities in the function $f(\mu)$, which seem to render the above calculations invalid.

Using these properties we can see that the infinite product of Gamma-functions in (40) vanishes and we end up with only a finite product of sine-functions. In the same way the overall scalar factors of all other S-matrix elements involving the scattering of breathers can be simplified into finite products, and we obtain the expressions (27) - (29) in section 4.

We will briefly show some of the steps in the calculation which lead to the expression (33) of the scalar factor $f_{p,r}(\mu)$. In order to obtain the commutation relations for the scattering of two excited solitons, e.g. A^j_p and A^j_r, we need to consider the quadruple product:

$$[\gamma_1\overline{A}^l(\theta_1)\overline{A}^k(\theta_2) + \gamma_2\overline{A}^k(\theta_1)\overline{A}^l(\theta_2)] \times [\gamma_1\overline{A}^l(\theta_3)\overline{A}^k(\theta_4) + \gamma_2\overline{A}^k(\theta_3)\overline{A}^l(\theta_4)] \tag{45}$$

Using the relations (15), we can work out this expression in a similar way as done above for $A^j\!-\!B_p$ scattering. Taking the 'double' limit: $\theta_2-\theta_1 \to -i\frac{2\pi}{3\lambda}p + i\frac{2\pi}{3}$ and $\theta_4-\theta_3 \to -i\frac{2\pi}{3\lambda}r + i\frac{2\pi}{3}$, we obtain the overall scalar factor in this product:

$$f_{0,0}(\mu+\frac{p+r}{2}-\lambda)f_{0,0}(\mu+\frac{r-p}{2})f_{0,0}(\mu+\frac{p-r}{2})f_{0,0}(\mu-\frac{p+r}{2}+\lambda).$$

By using the above properties (43) we can show that this expression is equal to

$$\frac{1}{(2i)^3}\frac{1}{\sin(\pi(\mu+\frac{p+r}{2}+2\lambda))\sin(\pi(\mu+\frac{p+r}{2}-\lambda))\sin(\pi(\mu+\frac{p+r}{2}))}f_{p,r}(\mu)$$

in which $\mu = i\frac{3\lambda}{2\pi}(\frac{\theta_1+\theta_2}{2}-\frac{\theta_3+\theta_4}{2})$ and $f_{p,r}(\mu)$ has the form (33) given in section 4. Considering all factors in the product (45) we finally obtain the exchange relations (30) in section 4.

APPENDIX D: Figures

(Time is moving upwards in all diagrams.)

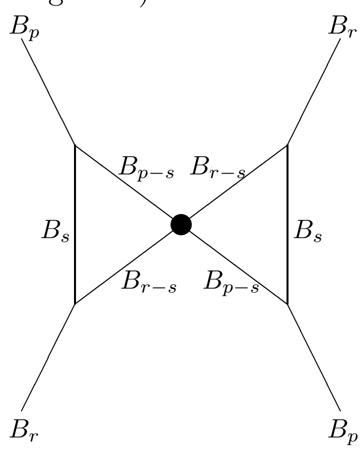

Figure 5a

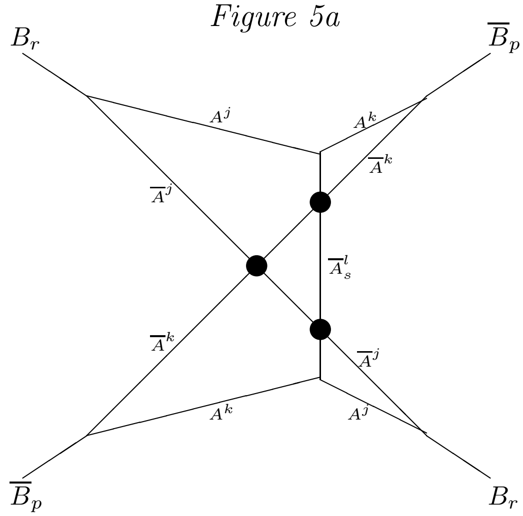

Figure 5b

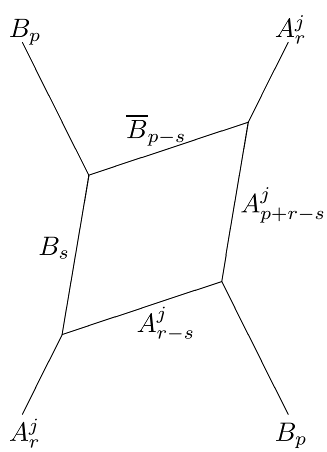

Figure 6

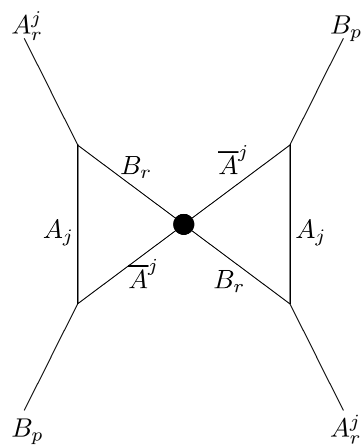

Figure 7a

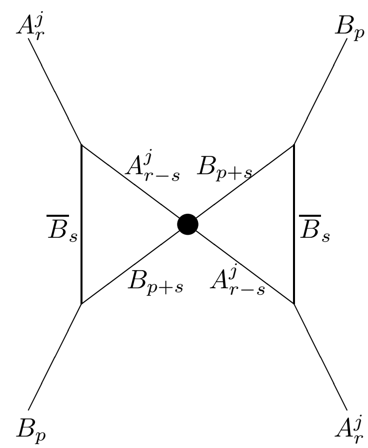

Figure 7b

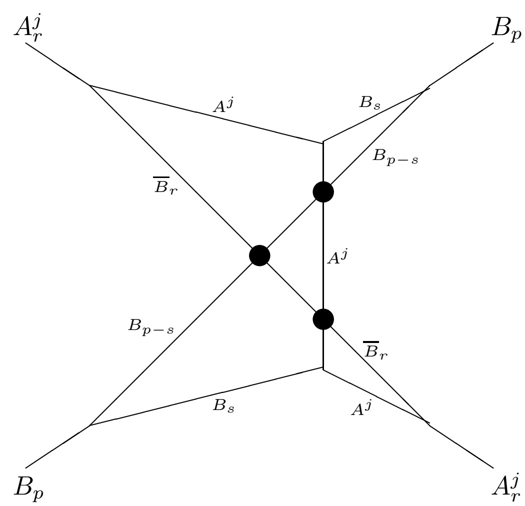

Figure 7c

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